A NEW CHARACTERIZATION OF COMMUTATIVE ARTINIAN RINGS

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Abstract. Let R be a commutative Noetherian ring. It is shown that R is Artinian if and only if every R-module is good, if and only if every R-module is representable. As a result, it follows that every nonzero submodule of any representable R-module is representable if and only if R is Artinian. This provides an answer to a question which is investigated in [**1**].

1. Introduction

All rings considered in this paper are assumed to be commutative with identity. There are several characterizations of Artinian rings. In particular, it is known that a Noetherian ring R is Artinian if and only if every prime ideal of R is maximal. In this article, we present a new characterization of Artinian rings according to the notions of primary decomposition and (its dual) secondary representation. To do so, we need to introduce a generalization of the notions Hopficity and co-Hopficity.

In [**2**] V. A. Hiremath introduced the concept of Hopficity for R-modules. The dual notion is defined by K. Varadarajan [**6**]. An R-module M is said to be Hopfian (resp. co-Hopfian) if any surjective (resp. injective) R-homomorphism is automatically an isomorphism. We refer the reader to [**6**] for reviewing the most important properties of Hopfian and co-Hopfian R-modules. We extend these definitions as follows: An R-module M is said to be semi Hopfian (resp. semi co-Hopfian) if for any $x \in R$, the endomorphism of M induced by multiplication by x is an isomorphism, provided it is surjective (resp. injective). Clearly any Hopfian (resp. co-Hopfian) R-module is semi-Hopfian (resp. semi co-Hopfian). Also, it is obvious that R, as an R-module, is Hopfian (resp. co-Hopfian) if and only if it is semi Hopfian (resp. semi co-Hopfian).

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As the main result of this note, we establish the following characterization of Artinian rings.

Theorem 1.1. *Let R be a commutative Noetherian ring. Then the following are equivalent:*
i) R is Artinian.
ii) Every nonzero R-module is good.
iii) Every R-module is semi Hopfian.
iv) Every nonzero R-module is representable.
iv') Every nonzero Noetherian R-module is representable.
v) Every R-module is semi co-Hopfian.
v') Every Noetherian R-module is semi co-Hopfian.
v") Every Noetherian R-module is co-Hopfian.
vi) Every nonzero R-module is Laskerian.

In [**1**] the following question was investigated: When are submodules of representable R-modules representable? In that paper [**1**, Theorem 2.3], it is shown that this is the case, when R is Von Neumann regular. For a Noetherian ring R, we prove that every nonzero submodules of any representable R-modules is representable if and only if R is Artinian (see 2.4).

2. The proof of the main theorem

Recall that a nonzero R-module M is called *good*, if its zero submodule possesses a primary decomposition. A nonzero R-module S is said to be *secondary*, if for any $x \in R$, the map induced by multiplication by x is either surjective or nilpotent. We say the R-module M is *representable*, if there are secondary submodules $S_1, S_2, \dots, S_k$ of M such that $M = S_1 + S_2 + \cdots + S_k$. The two notions primary decomposition and secondary representation are dual concepts. We refer the reader to [**3**, Appendix to §6], for more details about secondary representation. Also, recall that an R-module M is said to be *Laskerian*, if any submodule of M is an intersection of a finite number of primary submodules.

Lemma 2.1. *i) Every finitely generated R-module is Hopfian.*
ii) Every Artinian R-module is co-Hopfian.
iii) Every good R-module is semi Hopfian.
iv) Every representable R-module is semi co-Hopfian.

Proof. i) See [**7**, Proposition 1.2].
ii) is well known and can be checked easily.
iii) Let $x \in R$ be an M-coregular element of R. Let $0 = \bigcap_{i=1}^{n} Q_i$ be a primary decomposition of the zero submodule of M. Fix $1 \leq i \leq n$. Since Q_i is a proper submodule of M and $\frac{M}{Q_i} \xrightarrow{x} \frac{M}{Q_i}$ is either injective or nilpotent, it follows that x is $\frac{M}{Q_i}$-regular. Now, if $xm = 0$ for some element m in M, then for each i, it follows that $xm \in Q_i$ and so $m \in Q_i$. Hence $m = 0$, and so x in M-regular as required.
iv) is similar to (iii). □

Example 2.2. Let N be a nonzero co-Hopfian R-module. Set $M = \bigoplus_{i \in \mathbb{N}} N$. Then M is semi co-Hopfian, but it is not co-Hopfian. To this end define the R-homomorphism $\psi : M \longrightarrow M$ by $\psi(m_1, m_2, \dots) = (0, m_1, m_2, \dots)$ for all $(m_1, m_2, \dots) \in M$. Then ψ is injective, while it is not surjective.

Proof of theorem 1.1. $(i) \Rightarrow (ii)$ Let M be an R-module. Since R is Artinian, it is representable as an R-module. Hence $M \simeq \mathrm{Hom}_R(R, M)$ is good, by [**4**, Theorem 2.8]. The implications $(ii) \Rightarrow (iii)$, $(iv) \Rightarrow (v)$ and $(iv)' \Rightarrow (v)'$ follow, by 2.1.

Now we prove $(iii) \Rightarrow (v)$. Let M be an R-module and $D(\cdot) = \mathrm{Hom}_R(\cdot, E)$, where E is an injective cogenerator of R. Let $x \in R$ be such that the map $M \xrightarrow{x} M$ is injective. Then the map $D(M) \xrightarrow{x} D(M)$ is surjective and it is also injective, because $D(M)$ is semi Hopfian. But this implies that x is M-coregular, as the functor $D(\cdot)$ is faithfully exact. Hence M is semi co-Hopfian.

$(v)' \Rightarrow (i)$ Suppose the contrary and assume that $\mathfrak{p} \subset \mathfrak{m}$ is a strict containment of prime ideals of R. Let $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then x is $R/\mathfrak{p}$-regular, but it is not $R/\mathfrak{p}$-coregular. We achieved at a contradiction. Therefore every prime ideal of R is maximal and so R is Artinian.

Next, we prove $(i) \Rightarrow (iv)$. Since R is Artinian, $\mathrm{Max}(R)$ is finite. Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$. There are $\mathfrak{m}_i$-primary ideals $\mathfrak{a}_i$ of R such that $R \simeq \prod_{i=1}^{k} R/\mathfrak{a}_i$. Let $F = \bigoplus_{j \in J} R$ be an arbitrary free R-module. Set $S_i = \bigoplus_{j \in J} R/\mathfrak{a}_i$ for $i = 1, 2, \dots, k$. Then

$$F \simeq \bigoplus_{j \in J} (\prod_{i=1}^{k} R/\mathfrak{a}_i) \simeq \prod_{i=1}^{k} S_i = \bigoplus_{i=1}^{k} S_i.$$

It follows that for each $i = 1, 2, \dots, k$, the R-module S_i is $\mathfrak{m}_i$-secondary and hence F is representable. But any R-module is homeomorphic image of some free R-module

and so the conclusion follows. Note that one can check easily that any nonzero quotient of a representable R-module is also representable.

It follows from [**8**, Theorem] that the statements (i) and (v”) are equivalent. Let N be a proper submodule of an R-module M. Then N possesses a primary decomposition if and only if the R-module M/N is good. Thus (ii) and (vi) are equivalent. Now, because the implications $(iv) \Rightarrow (iv')$ and $(v) \Rightarrow (v')$ are clearly hold, the proof is complete. □

Corollary 2.3. *Let M be an R-module such that the ring $R/\operatorname{Ann}_R M$ is Artinian. Then M is both good and representable.*

Proof. Set $S = R/\operatorname{Ann}_R M$. Then M possesses the structure of an S-module in a natural way. A subset N of M is an R-submodule of M if and only if it is an S-submodule of M. Thus it is straightforward to see that M is good (resp. representable) as an R-module if and only if it is good (resp. representable) as an S-module. Now the conclusion follows by 1.1. □

Proposition 2.4. *Let R be a Noetherian ring. The following statements are equivalent:*
i) Every nonzero submodule of any representable R-module is representable.
ii) R is Artinian.

Proof. $(ii) \Rightarrow (i)$ is clear by 1.1.
$(i) \Rightarrow (ii)$ By [**5**], any nonzero injective module over a commutative Noetherian ring is representable. Since any R-module can be embedded in an injective R-module, it follows that all nonzero R-modules are representable. Therefore by the implication $(iv) \Rightarrow (i)$ of 1.1, it follows that R is Artinian. □

A commutative ring R is said to be Von Neumann regular, if for each element $a \in R$, there exists $b \in R$ such that $a = a^2 b$. In [**1**, Theorem 2.3], it is shown that over a commutative Von Neumann regular ring R every nonzero submodule of a representable R-module is representable. Since commutative Artinian rings are Noetherian, we can deduce the following result, by 2.4.

Corollary 2.5. *Let R be a commutative Von Neumann regular ring. Then R is Noetherian if and only if it is Artinian.*

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